Associating Finite Groups with Dessins d'Enfants

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Overview

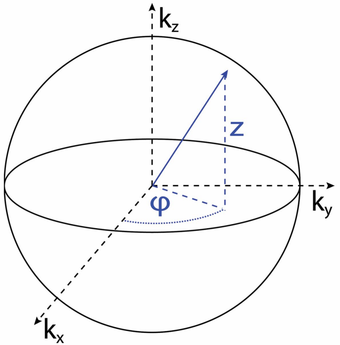
1. Stereographic Projection
2. Solids invariant under Aut$(\mathbb{P}^1(\mathbb{C}))$
3. Belyĭ maps/ Dessins d'Enfants
4. Platonic/Archimedean solids as Dessins d'Enfants
5. Some Johnson solids as Dessins d'Enfants

The Automorphisms of The Sphere

Definition

The real sphere of radius r is the surface

$$S^2(\mathbb{R}) = \left\{ (u, v, w) \in \mathbb{R}^3 \,\middle|\, u^2 + v^2 + w^2 = r^2 \right\}$$



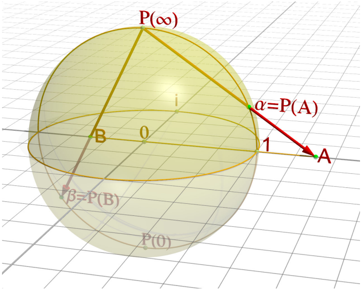
What are the automorphisms of the sphere?

1. Dilations / Contractions?
2. Translations?
3. Rotations?

We are only concerned with the **rigid rotations** of the **unit** sphere, that is, $r = 1$.

The Sphere as The Extended Complex Plane

Through stereographic projection, we can establish a bijection between the unit sphere $S^2(\mathbb{R})$ and the extended complex plane $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.



Definition

Define the stereographic projection from the unit sphere $S^2(\mathbb{R}) \xrightarrow{\sigma} \mathbb{P}^1(\mathbb{C})$:

$$\begin{array}{ccc} S^2(\mathbb{R}) & \xrightarrow{\sim} & \mathbb{P}^1(\mathbb{C}) \\ (u,\ v,\ w) & \mapsto & \dfrac{u + i\, v}{1 - w} \\ \left(\frac{2x}{x^2+y^2+1},\ \frac{2y}{x^2+y^2+1},\ \frac{x^2+y^2-1}{x^2+y^2+1} \right) & \leftarrow\!\shortmid & x + i\, y \end{array}$$

Automorphisms of The Sphere via Möbius Transformations

Definition

A Möbius Transformation of the extended complex plane $\mathbb{P}^1(\mathbb{C})$ is a rational function of the form

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

of one complex variable z with a, b, c, and d complex numbers satisfying $a\,d - b\,c \neq 0$. We denote the collection of such transformations by $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$.

1. The set $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ is a group under composition which maps $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$.
2. Any point (u, v, w) on the unit sphere corresponds to a complex number $z = x + i\,y$, there is a one-to-one correspondence with rigid rotations of the sphere and Möbius transformations.

Automorphism of The Sphere via Möbius Transformation

Theorem

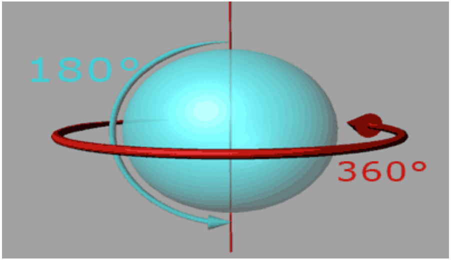
Let $SO_3(\mathbb{R})$ denote the group of rigid rotations $S^2(\mathbb{R}) \xrightarrow{\gamma} S^2(\mathbb{R})$ of the unit sphere, and let $\mathsf{Aut}(\mathbb{P}^1(\mathbb{C}))$ denote the group of Möbius transformations $\mathbb{P}^1(\mathbb{C}) \xrightarrow{f} \mathbb{P}^1(\mathbb{C})$. Then stereographic projection $S^2(\mathbb{R}) \xrightarrow{\sigma} \mathbb{P}^1(\mathbb{C})$ induces a bijection:

$$SO_3(\mathbb{R}) \xrightarrow{\sim} \mathsf{Aut}(\mathbb{P}^1(\mathbb{C}))$$

$$\begin{array}{ccc} \text{Sphere} & & \mathbb{C} \cup \{\infty\} \\ & & \\ S^2(\mathbb{R}) & \xrightarrow{\sigma} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow\gamma & & \downarrow f \\ S^2(\mathbb{R}) & \xleftarrow{\sigma^{-1}} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

Examples

1. Fix an angle θ. For $z \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, the function $f(z) = e^{i\theta} z$ is a Möbius transformation. Geometrically, this function represents a rotation along the w-axis via an angle θ
2. When $\theta = \frac{2\pi}{n}$ we rotate by a fraction of 2π. The Möbius transformation is $f(z) = \zeta_n \, z$, where $\zeta_n = e^{2\pi i/n}$ is an nth root of unity.
3. Similarly, the function $f(z) = 1/z$ is also a Möbius transformation. Geometrically, this function represents a flip along the v-axis.



Finite Automorphism Groups of The Sphere

Proposition (Felix Klein)

The following groups are finite subgroups of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$:

$$
\begin{array}{lll}
Z_n = \langle r | r^n = 1 \rangle : & r(z) = \zeta_n z & \\
D_n = \langle r, s | s^2 = r^n = (sr)^2 = 1 \rangle : & r(z) = \zeta_n z & s(z) = \dfrac{1}{z} \\
A_4 = \langle r, s | s^2 = r^3 = (sr)^3 = 1 \rangle : & r(z) = \dfrac{z + 2\,\zeta_3}{z - \zeta_3} & s(z) = \dfrac{z+2}{z-1} \\
S_4 = \langle r, s | s^2 = r^3 = (sr)^4 = 1 \rangle : & r(z) = \dfrac{z + \zeta_4}{z - \zeta_4} & s(z) = \dfrac{z+1}{z-1} \\
A_5 = \langle r, s | s^2 = r^3 = (sr)^5 = 1 \rangle : & r(z) = \dfrac{\varphi - {\zeta_5}^3\, z}{\varphi\, {\zeta_5}^3\, z + 1} & s(z) = \dfrac{\varphi - z}{\varphi\, z + 1}
\end{array}
$$

where $\zeta_n = e^{2\pi i/n}$ is a root of unity, and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Conversely, if G is a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$, then G is isomorphic to one of the five types of groups above.

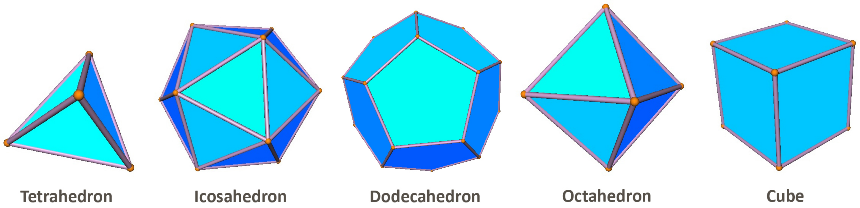
What Are The Solids Invariant Under The Finite Automorphism Groups ?

Solids Invariant Under Finite Automorphism Groups

Definition

1. A **Platonic solid** is a regular, convex polyhedron. Its faces are congruent, regular polygons, with the same number of faces meeting at each vertex. They are named after Plato (424 BC – 348 BC). There are 5 solids with these criteria.
2. An **Archimedean solid** is a highly symmetric, semi-regular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. Discovered by Johannes Kepler (1571 – 1630) in 1620, they are named after Archimedes (287 BC – 212 BC).There are 13 Archimedean solids.
3. A **Catalan solid** is a dual of an Archimedean solid. They are named after Eugéne Catalan (1814 – 1894) who discovered them in 1865. There are 13 Catalan solids.
4. A **Johnson solid** is a strictly convex polyhedron, each face of which is a regular polygon. They are named after Norman Johnson (1930 -) who discovered them in 1966. There are 92 Johnson solids.

Solids Invariant: The Platonic Solids

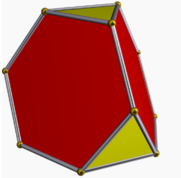


Rotation Groups

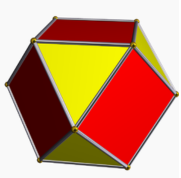
$A_4 = \langle r, s \mid s^2 = r^3 = (sr)^3 = 1 \rangle$	Tetrahedron
$A_5 = \langle r, s \mid s^2 = r^3 = (sr)^5 = 1 \rangle$	Icosahedron/ Dodecahedron
$S_4 = \langle r, s \mid s^2 = r^3 = (sr)^4 = 1 \rangle$	Octahedron/Cube

Solids Invariant: The Archimedean/Catalan Solids

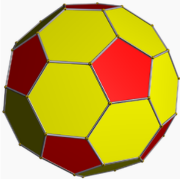
Truncated Tetrahedron



Cuboctahedron



Truncated Icosahedron

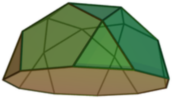


$$A_4 = \langle r, s | s^2 = r^3 = (sr)^3 = 1 \rangle$$
$$S_4 = \langle r, s | s^2 = r^3 = (sr)^4 = 1 \rangle$$
$$A_5 = \langle r, s | s^2 = r^3 = (sr)^5 = 1 \rangle$$

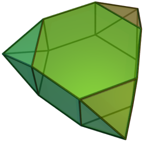
1. The Truncated Tetrahedron is the only Archimedean solid associated with A_4. The Triakis Tetrahedron is the only Catalan Solid associated with A_4.
2. They are 6 Archimedean solids and 6 Catalan solids associated with the rotations group S_4.
3. They are 6 Archimedean solids and 6 Catalan solids associated with the rotations group A_5.

Solids Invariant: The Johnson Solids

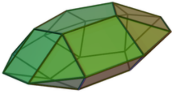
Pentagonal Rotunda



Triaugmented Hexagonal Prism



Pentagonal Bicopula



$Z_n = \langle r | r^n = 1 \rangle$

$D_n = \langle r, s | s^2 = r^n = (sr)^2 = 1 \rangle$

1. All 92 Johnson solids have either Cyclic symmetry or Dihedral symmetry.
2. Families of solids with cyclic symmetry: *Pyramids/Wheels, Rotundas, Cupolas, Elongated Pyramid, Gyroelongated Pyramid*.
3. Families of solids with dihedral symmetry: *Elongated Bypyramids, Truncated Bipyramids, Gyroelongated Bipyramids, Birotundas, Bicupolas, Truncated Trapezohedron, Dipoles/Hosohedron*.
4. There are some Johnson solids that do not fit into the aforementioned families of solids.

Can We Embed These Solids In The Riemann Sphere?

Belyĭ Maps

Definition

Fix a finite subgroup $G \subseteq \text{Aut}(\mathbb{P}^1(\mathbb{C}))$. A Belyĭ map $\mathbb{P}^1(\mathbb{C}) \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C})$ associated to G is a function satisfying the following:

1. It is a rational function, ie $\beta(z) = p(z)/q(z)$ for two relatively prime polynomials $p(z)$ and $q(z)$.

2. It has at most three critical values, which lie within $\{0, 1, \infty\}$:

$$\left\{ w \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; \begin{array}{c} w = \beta(z_0) \text{ for some } z_0 \in \mathbb{P}^1(\mathbb{C}) \\ \text{such that } \beta'(z_0) = 0 \end{array} \right\} \subseteq \{0, 1, \infty\}.$$

3. The function is invariant precisely under $G-$action:

$$\left\{ \gamma \in \text{Aut}(\mathbb{P}^1(\mathbb{C})) \;\middle|\; \beta(\gamma(z)) = \beta(z) \right\} = G.$$

Dessins d'Enfants

Definition

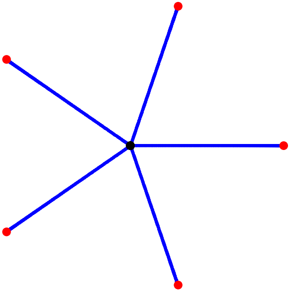
Given a Belyĭ map $\mathbb{P}^1(\mathbb{C}) \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C})$, a Dessin d'Enfants Δ_β is a connected, bipartite, planar graph with the following properties.

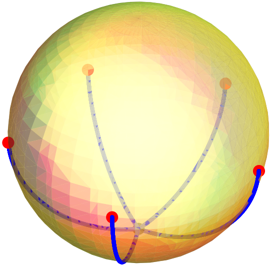
1. The "black" vertices are $B = \beta^{-1}(0)$.
2. The "white" vertices are $W = \beta^{-1}(1)$.
3. The edges are $E = \beta^{-1}([0,1])$.
4. The midpoints of the faces are $F = \beta^{-1}(\infty)$.

Remark: The phrase "Dessins d'Enfants" originated from Alexander Grothendieck. He viewed this construction as simple as "Children's Drawings." The graph Δ_β has symmetry reflected by G.

Example

Dessin for the Belyǐ function $\beta(z) = z^5$

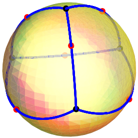




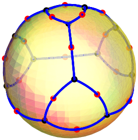
$\#\text{Vertices} = 6$ $\#\text{Edges} = 5$ $\#\text{Faces} = 1$

Can The Platonic, Archimidean, And Catalan Solids Be Realized As Dessins d'Enfants ?

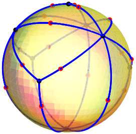
Solids As Dessins: Rotation Group A_4



- Tetrahedron
- Platonic Solid
- $\beta(z) = -\dfrac{64z^3(z^3-1)^3}{(8z^3+1)^3}$

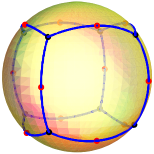


- Truncated Tetrahedron
- Archimedean Solid
- $\beta(z) = \dfrac{(1-232z^3+960z^6-256z^9+256z^{12})^3}{1728z^3(z^3-1)^3(8z^3+1)^6}$

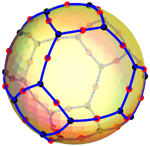


- Triakis Tetrahedron
- Catalan Solid
- $\beta(z) = \dfrac{1728z^3(z^3-1)^3(8z^3+1)^6}{(1-232z^3+960z^6-256z^9+256z^{12})^3}$

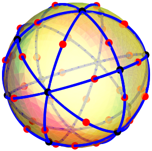
Solids As Dessins: Rotation Group S_4



- Cube
- Platonic Solid
- $\beta(z) = \dfrac{(1+14z^4+z^8)^3}{108z^4(-1+z^4)^4}$

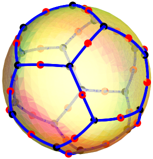


- Truncated Octahedron
- Archimedean Solid
- $\beta(z) = \dfrac{(1-390z^4+2319z^8+236z^{12}+2319z^{16}-390z^{20}+z^{24})^3}{2916z^4(-1+z^4)^4(1+14z^4+z^8)^6}$

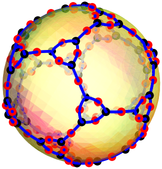


- Tetrakis Hexahedron
- Catalan Solid
- $\beta(z) = \dfrac{2916z^4(-1+z^4)^4(1+14z^4+z^8)^6}{(1-390z^4+2319z^8+236z^{12}+2319z^{16}-390z^{20}+z^{24})^3}$

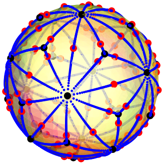
Solids As Dessins: Rotation Group A_5



- Dodecahedron
- Platonic Solid
- $\beta(z) = \dfrac{(1 - 228z^5 + 494z^{10} + 228z^{15} + z^{20})^3}{1728z^5(-1 - 11z^5 + z^{10})^5}$



- Truncated Dodecahedron
- Archimedean Solid
- $$\beta(z) = \frac{\begin{pmatrix} 1 - 252z^5 + 181194z^{10} - 12006900z^{15} + 83115375z^{20} \\ - 100628424z^{25} + 25004828z^{30} + 100628424z^{35} \\ + 83115375z^{40} + 12006900z^{45} + 181194z^{50} + 252z^{55} + z^{60} \end{pmatrix}^3}{1259712z^{10}(-1-11z^5+z^{10})^{10}(1-228z^5+494z^{10}+228z^{15}+z^{20})^3}$$



- Triakis Icosahedron
- Catalan Solid
- $$\beta(z) = \frac{1259712z^{10}(-1-11z^5+z^{10})^{10}(1-228z^5+494z^{10}+228z^{15}+z^{20})^3}{\begin{pmatrix} 1 - 252z^5 + 181194z^{10} - 12006900z^{15} + 83115375z^{20} \\ - 100628424z^{25} + 25004828z^{30} + 100628424z^{35} \\ + 83115375z^{40} + 12006900z^{45} + 181194z^{50} + 252z^{55} + z^{60} \end{pmatrix}^3}$$

How Did We Find Such Belyĭ Maps ?

Belyĭ Maps: Platonic Solids

Proposition (Felix Klein, 1875)

Let Γ denote the vertices of a Platonic solid. Then there exists a Belyĭ map $\mathbb{P}^1(\mathbb{C}) \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C})$ such that $\Gamma \simeq \Delta_\beta$ is the Dessin d'Enfants of β.

$$\beta_{\text{tetrahedron}}(z) = \frac{64(z^3-1)^3}{z^3(z^3+8)^3}$$

$$\beta_{\text{cube}}(z) = \frac{(1+14z^4+z^8)^3}{108z^4(-1+z^4)^4}$$

$$\beta_{\text{octahedron}}(z) = \frac{1}{\beta_{\text{cube}}(z)}$$

$$\beta_{\text{dodecahedron}}(z) = \frac{(1-228z^5+494z^{10}+228z^{15}+z^{20})^3}{1728z^5(-1-11z^5+z^{10})^5}$$

$$\beta_{\text{icosahedron}}(z) = \frac{1}{\beta_{\text{dodecahedron}}(z)}$$

Belyĭ Maps: Platonic Solids

Klein's Approach

1. Embed the vertices of a Platonic solid in the unit sphere, then use Stereographic Projection to write them as complex numbers.

2. Find a homogeneous polynomial which vanishes at these vertices.

3. Use Invariant Theory to list three more polynomials c_4, c_6, and Δ with a syzygy among them.

4. Define the Belyĭ map as:

$$\beta(z) = \frac{c_4(z)^3 - c_6(z)^2}{c_4(z)^3}$$

Belyĭ maps: Tetrahedron Example

- Let $B = \{P_\infty, P_0, P_1, P_2\} \subseteq S^2(\mathbb{R})$ denote the four vertices of a **tetrahedron** embedded into the Riemann sphere. Explicitly:

$$\begin{aligned} P_\infty &= (0,0,1) \\ P_k &= \left(\frac{2\sqrt{2}}{3} \cos \frac{2\pi k}{3},\ \frac{2\sqrt{2}}{3} \sin \frac{2\pi k}{3},\ -\frac{1}{3} \right) \qquad k = 0,1,2. \end{aligned}$$

  Mapping them through σ, we obtain images of four points in $\mathbb{P}^1(\mathbb{C})$.

$$\sigma(B) = \{\infty,\ \zeta_3,\ \zeta_3^2,\ 1\},$$

- A **homogeneous polynomial** which vanishes on the vertices $\sigma(B) \subseteq \mathbb{P}^1(\mathbb{C})$ is given by,

$$\delta(\tau_1, \tau_0) = 3\tau_0(\tau_1^3 - \tau_0^3).$$

Example

- **Using Invariant Theory, we list three more homogeneous polynomials**

$$c_4(\tau_1,\tau_0) = (\text{constant}) \cdot Hess(\delta)(\tau_1,\tau_0) = 9\tau_1(\tau_1^3 + 8\tau_0^3)$$

$$c_6(\tau_1,\tau_0) = (\text{constant}) \cdot Cov(\delta, c_4)(\tau_1,\tau_0) = 27(\tau_1^6 - 20\tau_1^3\tau_0^3 - 8\tau_0^6)$$

$$\Delta(\tau_1,\tau_0) = \delta(\tau_1,\tau_0)^3 = 27\tau_0^3(\tau_1^3 - \tau_0^3)^3$$

Where

$$Hess(\delta)(\tau_1,\tau_0) = \frac{\partial^2\delta}{\partial\tau_1^2} \cdot \frac{\partial^2\delta}{\partial\tau_0^2} - \left(\frac{\partial^2\delta}{\partial\tau_1\partial\tau_0}\right)^2$$

$$Cov(\delta, c_4) = \frac{\partial\delta}{\partial\tau_1} \cdot \frac{\partial c_4}{\partial\tau_0} - \frac{\partial\delta}{\partial\tau_1} \cdot \frac{\partial c_4}{\partial\tau_0}$$

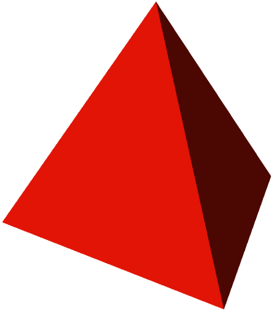
The syzygy relation among the three polynomials:

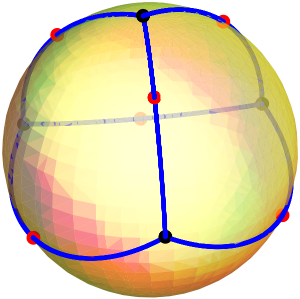
$$c_4(\tau_1,\tau_0)^3 - c_6(\tau_1,\tau_0)^2 = 1728\ \Delta(\tau_1,\tau_0)$$

Example

- The Belyĭ map for the Tetrahedron is

$$\beta(z) = \frac{c_4(\tau_1,\tau_0)^3 - c_6(\tau_1,\tau_0)^2}{c_4(\tau_1,\tau_0)^3} = \frac{64(z^3-1)^3}{z^3(z^3+8)^3} \quad \text{where} \quad z = \frac{\tau_1}{\tau_0}$$





Belyĭ Maps: Archimedean and Catalan Solids

Proposition

The 13 Archimedean Solids and the 13 Catalan Solids can be derived from the 5 Platonic solids by 7 geometric operations:

- Truncation
- Bitruncation
- Rectification
- Birectification
- Snubification
- Rhombification
- Rhombitruncation

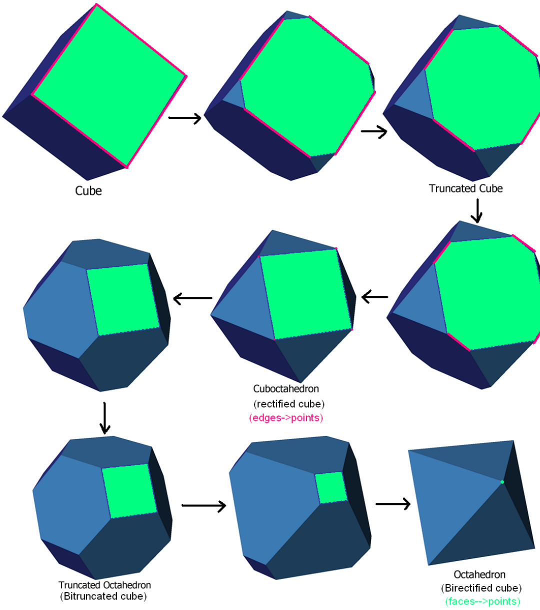
Proposition (N. Magot and A. Zvonkin, 2001)

These seven operations can be algebraically recognized as Belyĭ maps $\mathbb{P}^1(\mathbb{C}) \xrightarrow{\phi} \mathbb{P}^1(\mathbb{C})$. In particular, all of the Archimedean and Catalan solids can be realized as Dessins d'Enfants.

Geometric Operations: Definitions

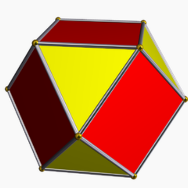
1. **Truncation**: a face in place of each vertex (vertices $\rightarrow$ faces)
2. **Rectification**: truncation at the midpoints of all edges (edges $\rightarrow$ vertices)
3. **Birectification**: faces $\leftrightarrow$ vertices
4. **Bitruncation**: truncation after birectification
5. **Rhombitruncation**: a truncation after rectification
6. **Rhombification**: a rectification after rectification
7. **Snubification**: "alternation" after truncation. ("Alternating" is the process of removing opposites vertices.

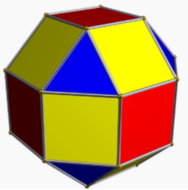
Geometric Operations: Examples



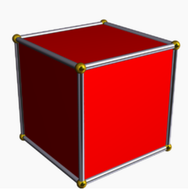
Geometric Operations: Examples

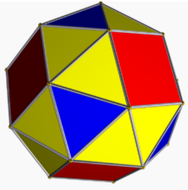
- **Rhombification**: Cuboctahedron → Rhombicuboctahedron





- **Snubification**: Cube → Snub Cube



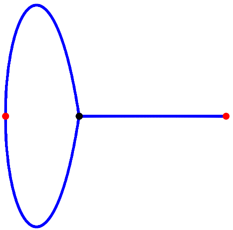


Approach (N. Magot and A. Zvonkin, 2001)

1. Determine the hypermap corresponding to a given geometric operation.
2. Deduce the Belyĭ map of this operation.
3. Compose this new function with a Platonic solid's Belyĭ map to get an Archimedean solid's Belyĭ map.
4. The Belyĭ map of a Catalan solid is the reciprocal of the corresponding Archimedean solid's Belyĭ map.

Example

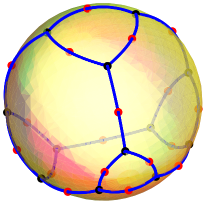
1. Hypermap of Truncation



2. Corresponding Belyĭ map

$$\phi_{\text{truncation}}(w) = \frac{(4\,w-1)^3}{27\,w}$$

3. Truncated Tetrahedron Belyĭ map



$$\beta = \phi_{\text{truncation}} \circ \beta_{\text{tetrahedron}}$$

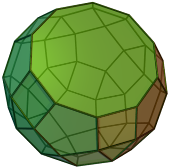
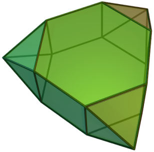
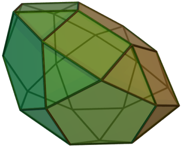
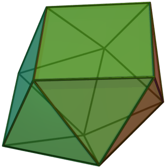
$$\beta(z) = \frac{(1-232z^3+960z^6-256z^9+256z^{12})^3}{1728z^3(z^3-1)^3(8z^3+1)^6}$$

Can We Realize Other Solids As Dessins?

The Johnson Solids

Definition

A Johnson Solid is a convex polyhedron with regular polygons as faces but which is not a Platonic or Archimedean.

Proposition

There are 92 distinct Johnson solids. All Johnson Solids have rotational symmetry groups isomorphic to either the cyclic group Z_n or the dihedral group D_n.

The Johnson Solids

Proposition

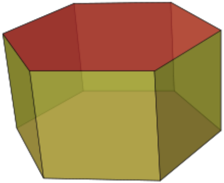
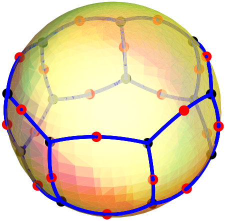
Most of the 92 Johnson Solids can be realized via "operations" on:

- Platonic Solids
- Archimedean and Catalan Solids
- Prisms and Antiprisms
- Cupolae
- Pyramids
- Rotunda

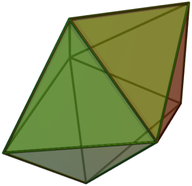
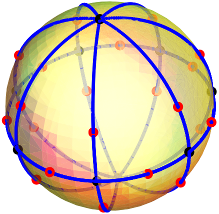
These six operations are:

- **Bi**: to take two copies of the solid and join them base-to-base.
- **Elongate**: to attach a prism to the base of the solid.
- **Gyroelongate**: to attach an antiprism to the base of the solid.
- **Augment**: to join a pyramid or cupola to a face.
- **Diminish**: to remove a pyramid or cupola from the solid.
- **Gyrate**: to take a cupola on the solid and rotate it such that different edges match up.

Prisms and Bipyramid

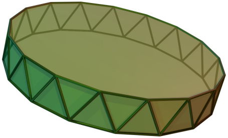



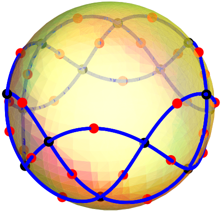
- Prism
- Rotation Group D_n
- $\beta(z) = \dfrac{(z^{2n} + 14\, z^n + 1)^3}{108\, z^n\, (z^n - 1)^4}$

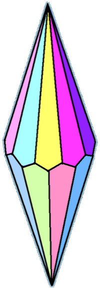
- Bipyramid
- Rotation Group D_n
- $\beta(z) = \dfrac{108\, z^n\, (z^n - 1)^4}{(z^{2n} + 14\, z^n + 1)^3}$

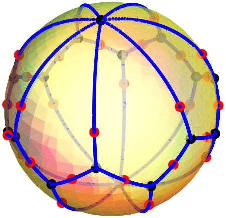
Antiprisms and Trapezohedron





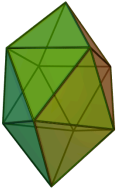
- Antiprism
- Rotation Group D_n
- $\beta(z) =$
$$-\frac{(8\,z^{2n} - 20\,z^n - 1)^4}{256\,z^n\,(z^n+1)^3\,(8\,z^n-1)^3}$$

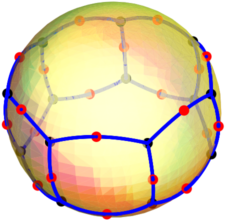




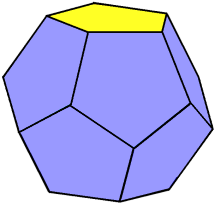
- Trapezohedron
- Rotation Group D_n
- $\beta(z) =$
$$-\frac{256\,z^n\,(z^n+1)^3\,(8\,z^n-1)^3}{(8\,z^{2n} - 20\,z^n - 1)^4}$$

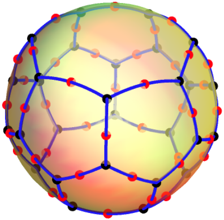
New Results



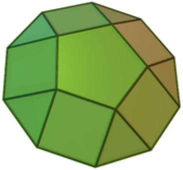
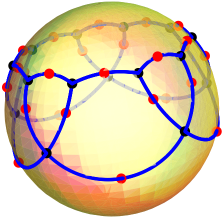


- Gyroelongated Bipyramid
- Rotation Group D_n
- $\beta(z) =$
$$\frac{1728\, z^n\, (z^{2n} - 11\, z^n - 1)^5}{(z^{4n} + 228\, z^{3n} + 494\, z^{2n} - 228\, z^n + 1)^3}$$

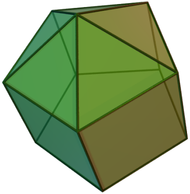
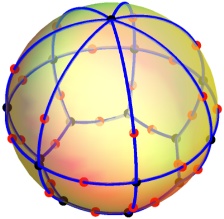




- Truncated Trapezohedron
- Rotation Group D_n
- $\beta(z) =$
$$\frac{(z^{4n} + 228\, z^{3n} + 494\, z^{2n} - 228\, z^n + 1)^3}{1728\, z^n\, (z^{2n} - 11\, z^n - 1)^5}$$

- Cupola
- Rotation Group Z_n
- $\beta(z) = \dfrac{27\,(z^n - 1)^4\,(3\,z^{2n} - 16\,z^n + 1728)^3}{4\,z^n\,(5\,z^n - 54)^3\,(9\,z^n + 40)^4}$

- Elongated Pyramid
- Rotation Group Z_n
- $\beta(z) = 4\,(-665857 + 470832\,\sqrt{2})\,\cdot$
  $$\frac{z^n\,(z^n-1)^4\,\left[z^n-4\,(41+29\,\sqrt{2})\right]^3}{\left[(-24+17\,\sqrt{2})\,z^n+1\right]^4\,\left[4\,(2+\sqrt{2})\,z^n+1\right]^3}$$

Future Work

Meta-goal: Find Belyĭ maps in order to realize all the Johnson solids as a Dessin d'Enfants.

- Partial result: we have found the Belyĭ maps β for all the building blocks of Johnson solids.
- Write down factorizations $\beta' = \phi \circ \beta$ of the Belyĭ maps β' for all the Johnson solids in terms of the Belyĭ maps β for our building blocks and functions ϕ associated to the operations.

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